Abelianess implies quasi-affiness revisited

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outline

1. Old stuff
   - Abelian algebras
   - Affine algebras
   - Quasi-affine algebras

2. New stuff
   - Toolbox
   - Subreducts of modules

abelian algebras

An algebra $\mathbb{A}$ is abelian if the diagonal $D_{\mathbb{A}} = \{(a, a) | a \in \mathbb{A}\}$ is a class of a congruence of $\mathbb{A}^2$.

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$$t(a,\underline{u}) = t(a,\underline{v}) \quad \Leftrightarrow \quad t(b,\underline{u}) = t(b,\underline{v}).$$

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Modules, unary algebras.

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Aim

Understand abelian algebras.

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What is a non locally finite generalization of Hobby-McKenzie theorem?

quasi-affine algebras

Recall

A non locally finite analog of omitting type **1** is the satisfaction of a nontrivial idempotent Mal'cev condition, .

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(A, Ω) is a reduct of (A, Φ) if each $\omega \in \Omega$ is a term operation of (A, Φ)
subreduct = subalgebra of a reduct
quasi-affine algebra = subreduct of an affine algebra

abelian $\Rightarrow$ quasi-affine

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Theorems (K. Kearnes and Á. Szendrei '98)

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The proof is based on R. Quackenbush's characterization of quasi-affine algebras '85.

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Today dish = a new "proof" of the first part of Kearnes-Szendrei theorem.

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Theorem (J. Ježek '79)

Each algebra $\mathbb{A} = (A, \Omega)$ is a subreduct of $F_{\mathcal{SM}}(A)/\theta_{\mathbb{A}}$.

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$\theta_{\mathbb{A}}$ - naturally constructed congruence

subreducts of modules

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$\mathbb{A}$ is a subreduct of a module iff $\theta_{\mathbb{A}}^{+} \cap A^2 = 0_A$.

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$\mathcal{Q}$ - set of all quasi-identities

$$[t_1 \approx s_1 \wedge \cdots \wedge t_n \approx s_n] \rightarrow t_0 \approx s_0,$$

such that

$$s_0 + \cdots + s_n = t_0 + \cdots + t_n.$$

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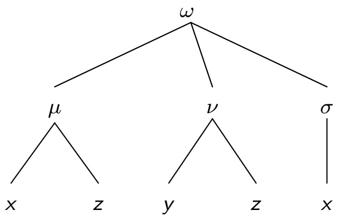
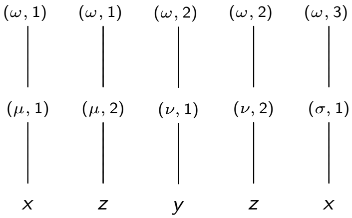
$\mathbb{A}$ is a subreduct of a module iff it satisfies $\mathcal{Q}$ iff it is quasi-affine.

$\mathcal{Q}$ once more

Branch decomposition of terms



$t(x, z)$



$\mathrm{BD}(t)$

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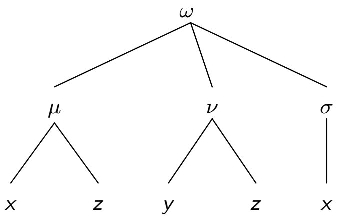

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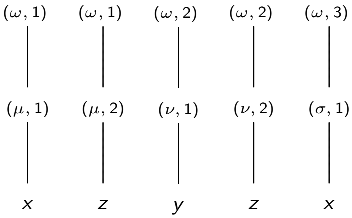

BD(t)

Fact

A quasi-identity $[t_1 \approx s_1 \wedge \cdots \wedge t_n \approx s_n] \rightarrow t_0 \approx s_0$ belongs to $\mathcal{Q}$ iff the following equality of multisets is valid

$$\biguplus_{i=0}^{n} \mathrm{BD}(t_i) = \biguplus_{i=0}^{n} \mathrm{BD}(s_i).$$

abelian $\Rightarrow$ subreduct of a module

Corollary

Let $\mathbb{A}$ be an algebra. Assume that for each $k \in \mathbb{Z}^+$ there is a binary relation $\int$ of A^k such that

- $\underline{a} \int \sigma \underline{a}$ for each permutation $\sigma \in [k]!$,
- $(\underline{c}, t(a, \underline{d}), \underline{e}) \int (\underline{c}', t(a, \underline{d}'), \underline{e}') \;\;\Leftrightarrow\;\; (\underline{c}, t(b, \underline{d}), \underline{e}) \int (\underline{c}', t(b, \underline{d}'), \underline{e}')$
- $(a, \underline{c}) \int (b, \underline{c}) \;\;\Rightarrow\;\; a = b$.

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Proof of Kearnes-Szendrei theorem

If $\mathbb{A}$ is abelian and $\mathcal{V}(\mathbb{A})$ satisfies nontrivial IMC, then we may construct $\int$ from the largest congruence of $\mathbb{A}^2$ such that the diagonal $D_{\mathbb{A}}$ is its class.

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